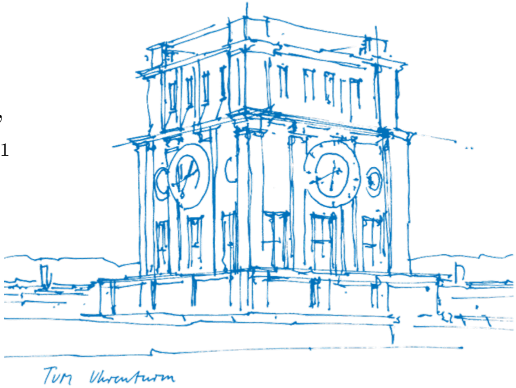
Generic Decoding of Restricted Errors

Sebastian Bitzer[1], Alessio Pavoni[2], Violetta Weger[1], Paolo Santini[2], Marco Baldi[2], Antonia Wachter-Zeh[1]

[1]Technical University of Munich

[2]Università Politecnica delle Marche

June 26, 2023



Outline

The Restricted Syndrome Decoding Problem

Restricted Syndrome Decoding Problem (R-SDP)

Given: parity-check matrix $\boldsymbol{H} \in \mathbb{F}_p^{(n-k)\times n}$, syndrome $\boldsymbol{s} \in \mathbb{F}_p^{n-k}$, weight t, $g \in \mathbb{F}_p$ of order z and $\mathbb{E} = \{g^0, \dots, g^{z-1}\} \subset \mathbb{F}_p^*$.

Find: error $\boldsymbol{e} \in (\mathbb{E} \cup \{0\})^n$ such that $\boldsymbol{H}\boldsymbol{e}^\top = \boldsymbol{s}$ and $\mathrm{wt}(\boldsymbol{e}) = t$.

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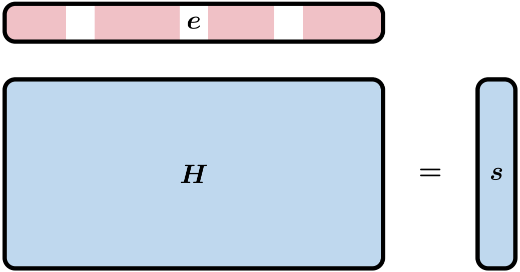
Improved solvers using the representation technique[1]

[1] Howgrave-Graham, N., & Joux, A. (2010). New generic algorithms for hard knapsacks. *Eurocrypt*

The General Framework



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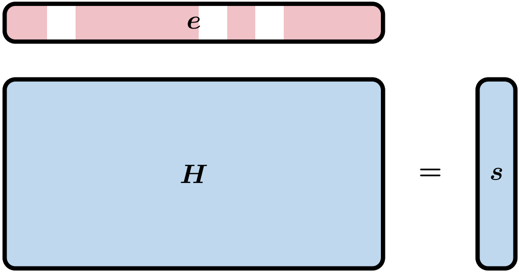

1. Random permutation

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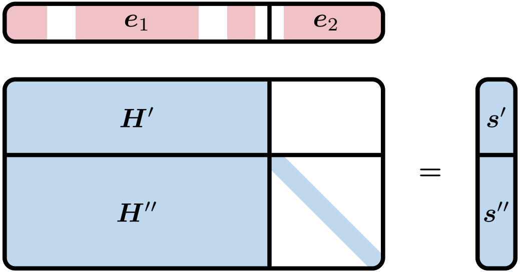

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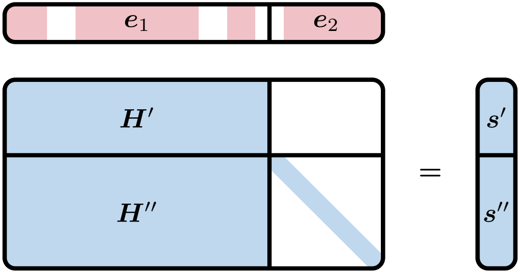

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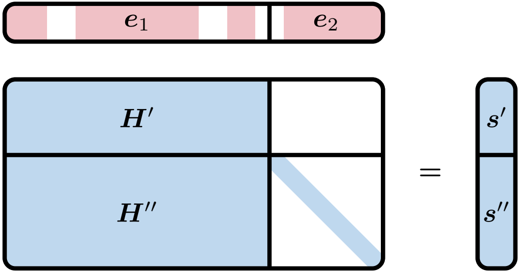

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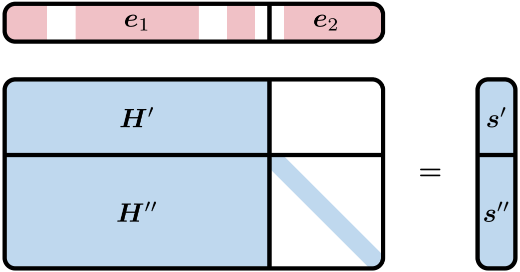

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$\Rightarrow$ cost $= \frac{\text{enumeration cost}}{\text{success probability}}$

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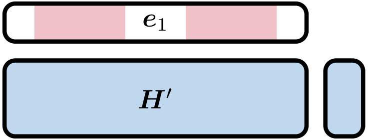
A Meet-in-the-Middle Strategy



A Meet-in-the-Middle Strategy

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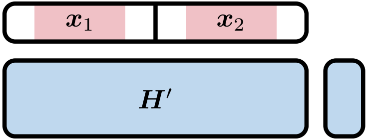

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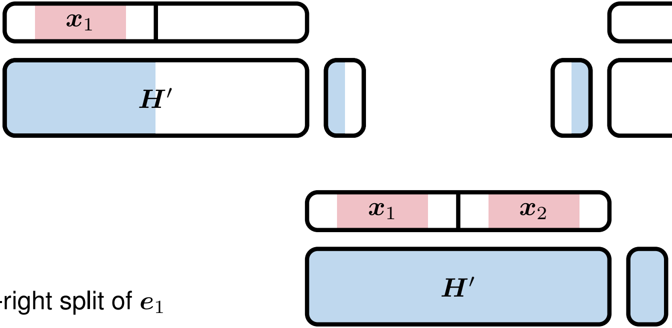

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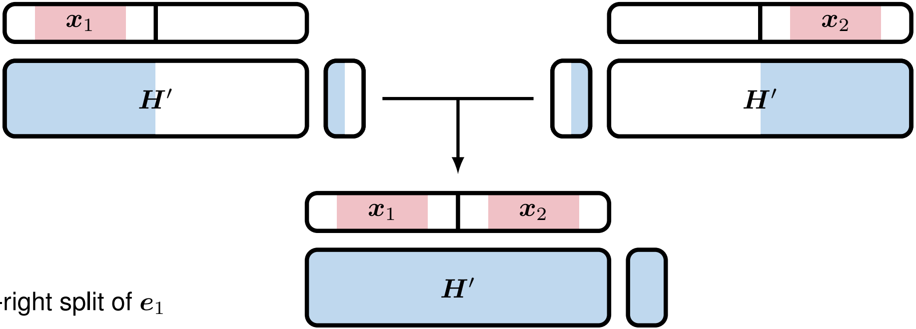
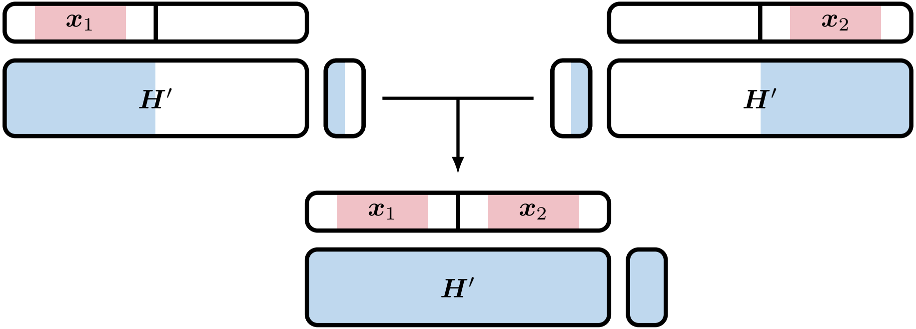


- Left-right split of e_1
- Enumerate x_1, x_2
- Collisions solve small instance

The Representation Technique



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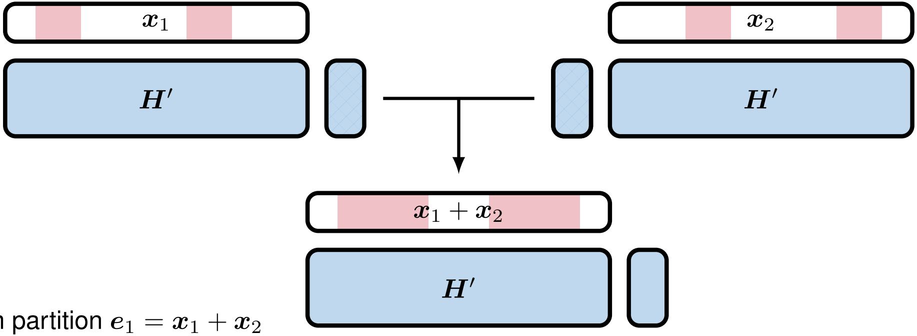

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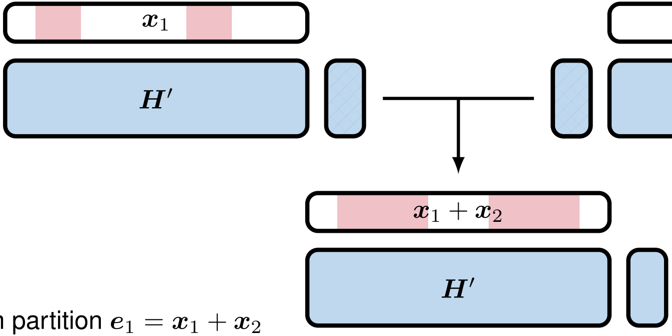

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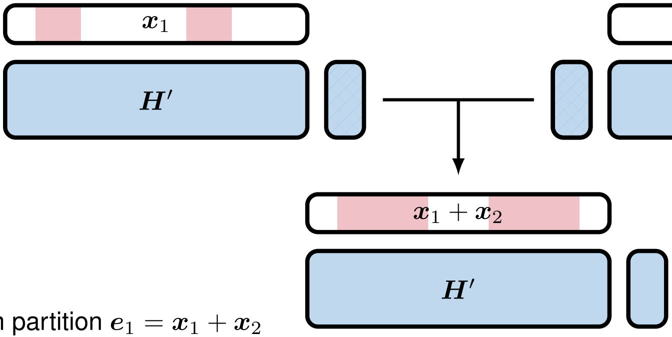

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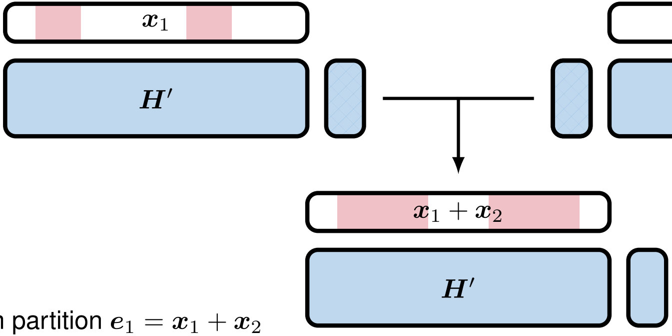

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Search space for x_1, x_2?

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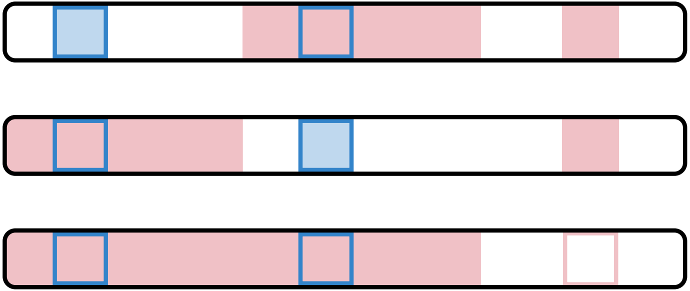

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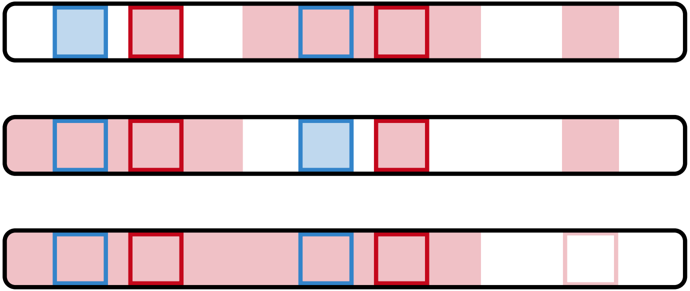

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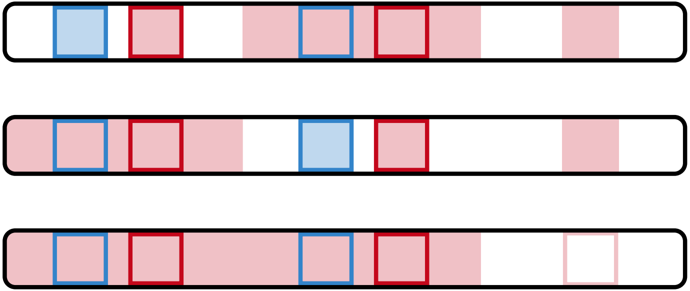

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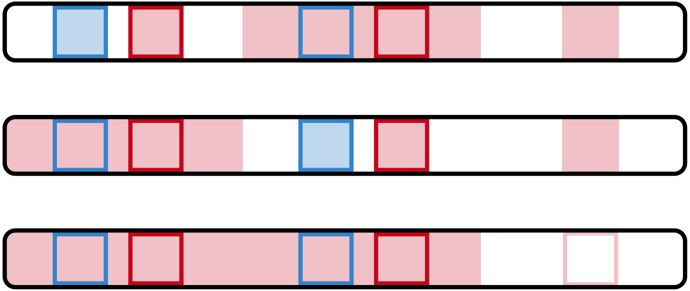

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Performance highly dependent on structure of $\mathbb{E}$

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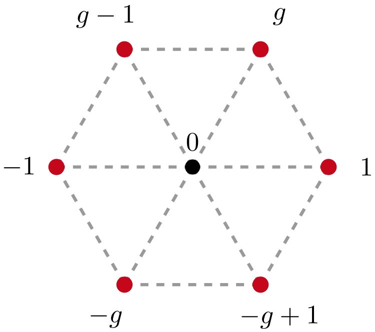

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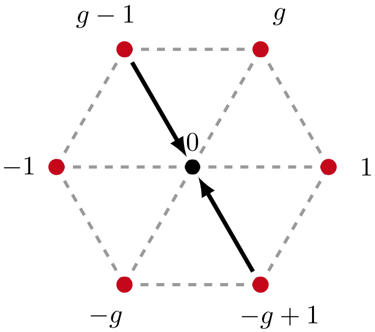

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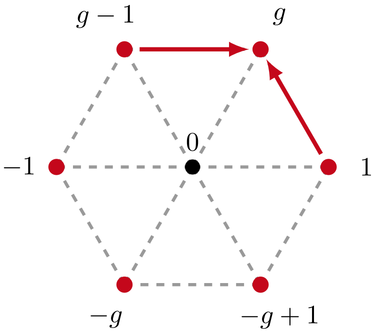

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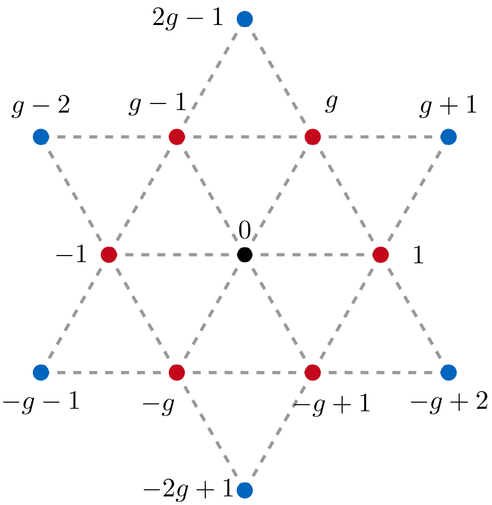

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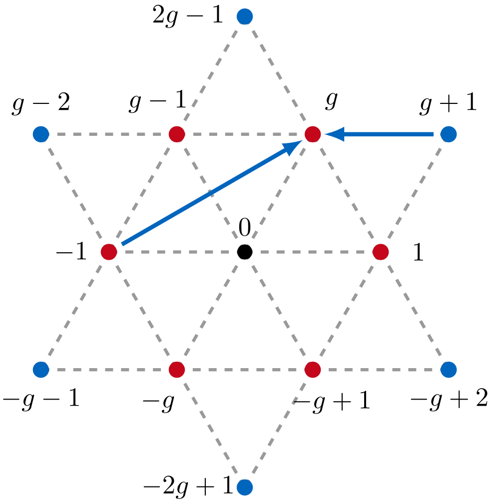

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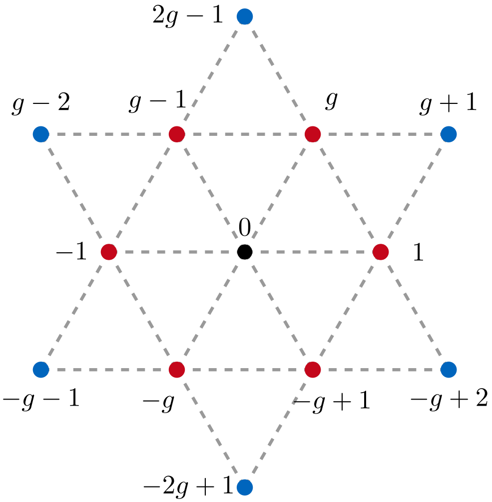

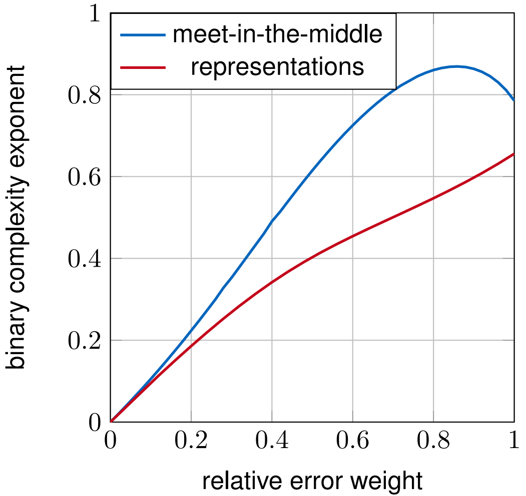

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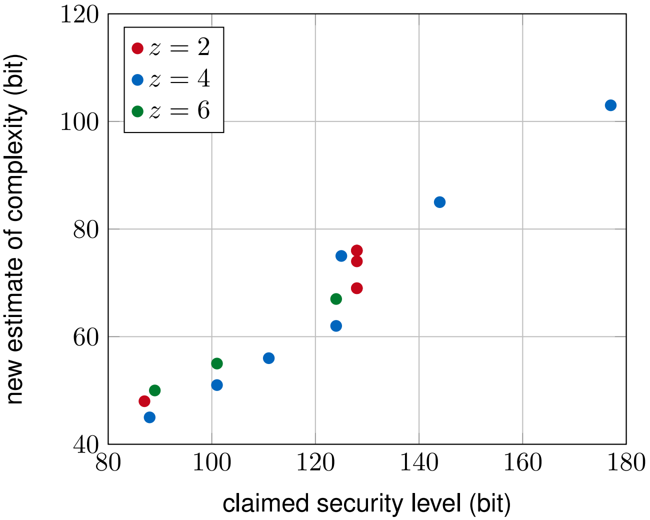
Overview of Results



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CROSS

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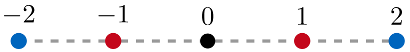
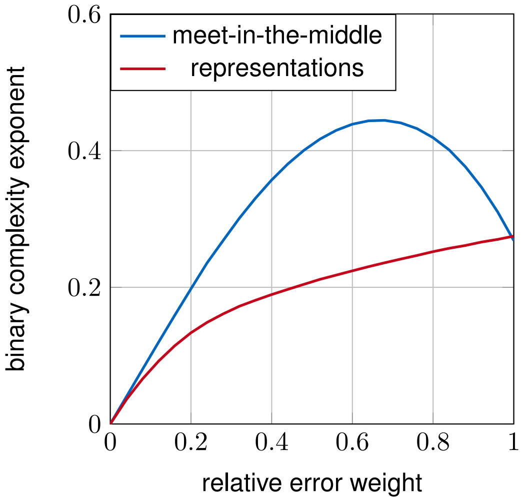
Thank you! Questions?

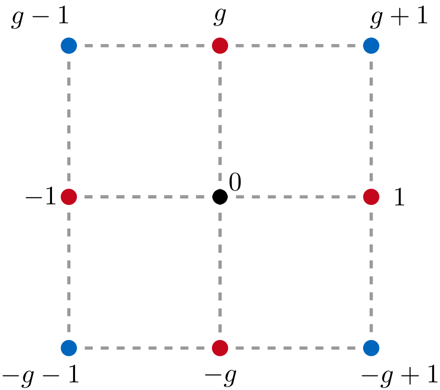
$z = 2$: Simple Structure[6]

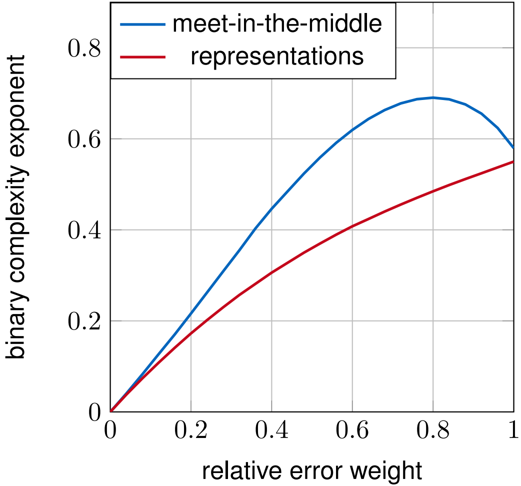




[6] Baldi, M., Chiaraluce, F., & Santini, P. (2021). Code-based signatures without trapdoors through restricted vectors. *Cryptology ePrint Archive*

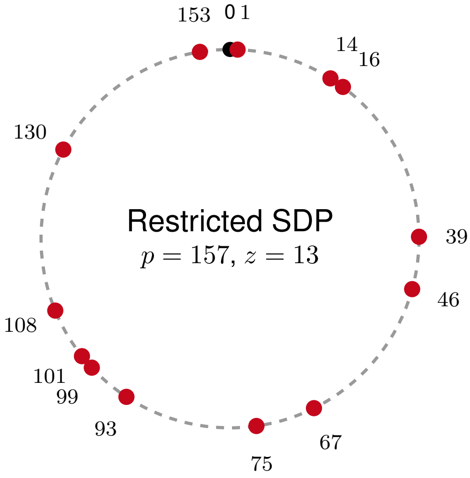
$z = 4$: Gaussian Integers[7]

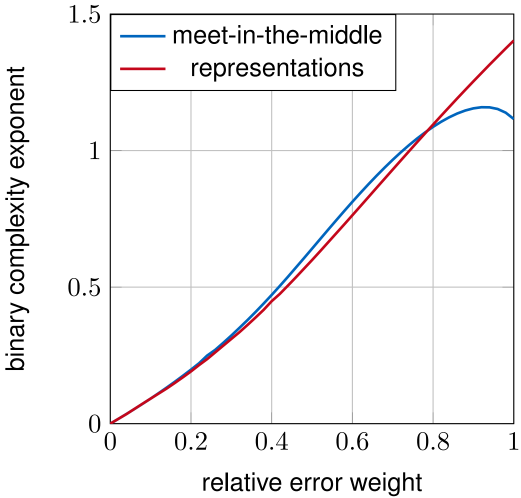




[7] Freudenberger, J., & Thiers, J.-P. (2021). A new class of q-ary codes for the McEliece cryptosystem. *Cryptography*

$z = 13$: Few Symmetries[8]





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