Weighted-Hamming Metric for Parallel Channels

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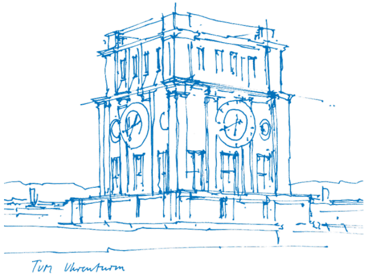

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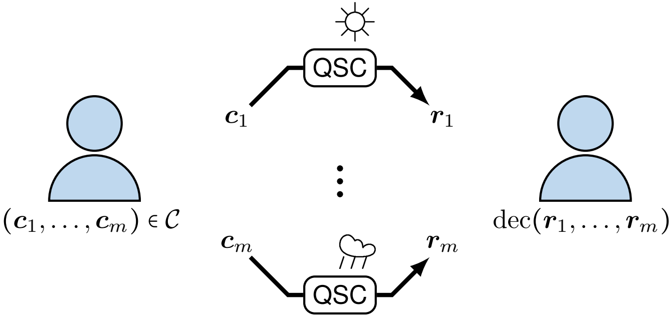

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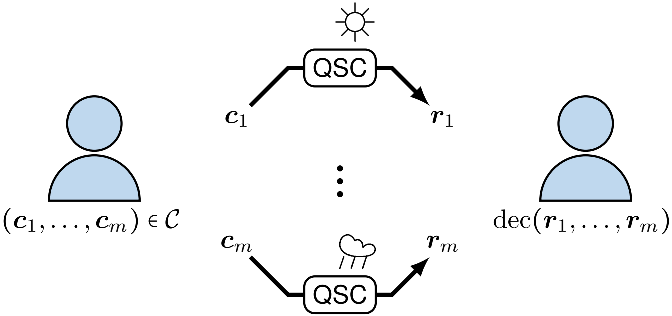

Standard solutions:

- Raptor
- Turbo
- LDPC
- Polar
- ...

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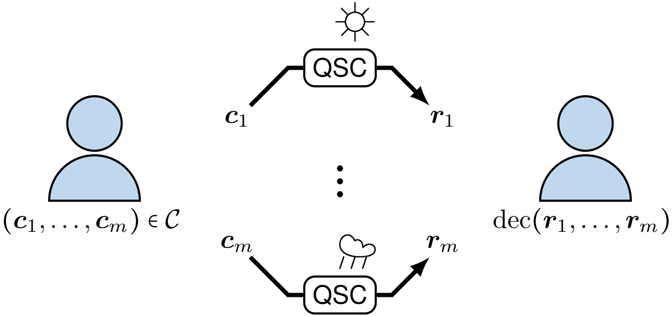

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Goal of this Work

Analyze codes that uniquely decode *all* errors with $P(e) \geq \theta$

From Probability to Metric

Matching a Channel ♡

A distance matches a channel if

$$d(\boldsymbol{c}, \boldsymbol{r}) \leq d(\boldsymbol{c}', \boldsymbol{r}) \iff P(\boldsymbol{c} \mid \boldsymbol{r}) \geq P(\boldsymbol{c}' \mid \boldsymbol{r})$$

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Weighted-Hamming Metric

Fix $(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$.

$$\mathrm{wt}(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m) = \sum_{\ell=1}^{m} \lambda_\ell \cdot \mathrm{wt}_{\mathrm{H}}(\boldsymbol{c}_\ell)$$

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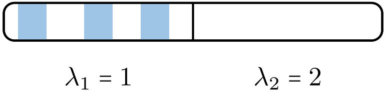

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$\lambda_1 = 1$ $\quad$ $\lambda_2 = 2$

$$\mathrm{wt_H}(\boldsymbol{c}_1, \boldsymbol{c}_2) = 3$$

$$\mathrm{wt}(\boldsymbol{c}_1, \boldsymbol{c}_2) = 6$$

Codes in the Weighted-Hamming Metric

Trivial constructions: → Hamming-metric code
→ Independent codes for subchannels

suboptimal

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Previous work:

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Open: General bounds and constructions

Error-Correction Capability

- Silva, Kschischang (2009). On Metrics for Error Correction in Network Coding.
- → General framework for adversarial error correction

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Error-Correction Capability

Let $\tau(\boldsymbol{c}) = \min_{\boldsymbol{r} \in \mathbb{F}_q^n} \max\{\mathrm{wt}(\boldsymbol{r}), \mathrm{wt}(\boldsymbol{c} - \boldsymbol{r})\} - 1$

Linear $\mathcal{C}$ t-error-correcting $\iff t \leq \tau(\mathcal{C}) = \min_{\boldsymbol{c} \in \mathcal{C} \setminus \{0\}} \tau(\boldsymbol{c})$

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Minimum Distance

$$\tau(\mathcal{C}) \ge \left\lfloor \frac{d(\mathcal{C}) - 1}{2} \right\rfloor$$

Equality for *normal* metrics

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Hamming metric, rank metric, . . . are normal.

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$$\tau(\mathcal{C}) \geq \left\lfloor \frac{d(\mathcal{C}) - 1}{2} \right\rfloor$$

Equality for *normal* metrics

Hamming metric, rank metric, ... are normal.

What about the weighted-Hamming metric?

Being Normal is Boring!

$$\lambda_1 = 1 \qquad \lambda_2 = 2$$

$$\boldsymbol{c}_0 = (0, 0, 0 \mid 0, 0, 0)$$

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Bound on τ via d

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Singleton-Like Upper Bound

Singleton-like Bound

Let $\lambda_1 \leq \ldots \leq \lambda_m$, $\dim(\mathcal{C}) = k$.

$$d \leq \sum_{\ell=1}^{\ell'} n_\ell \lambda_\ell + \left(\sum_{\ell=\ell'+1}^{m} n_\ell - k + 1 \right) \cdot \lambda_{\ell'+1},$$

where ℓ' maximal s.t. $\sum_{\ell=1}^{\ell'} n_\ell \leq n - k + 1$.

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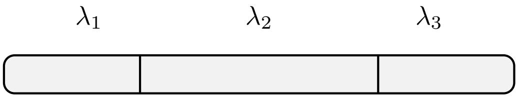

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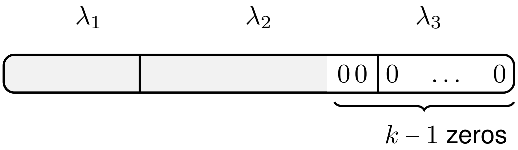

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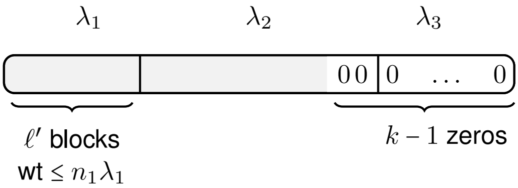

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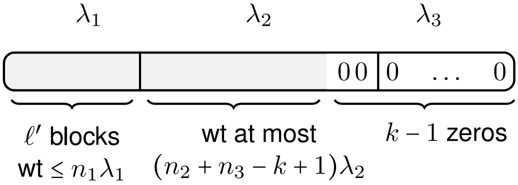

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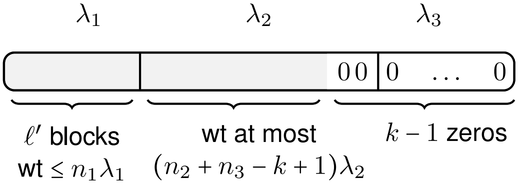
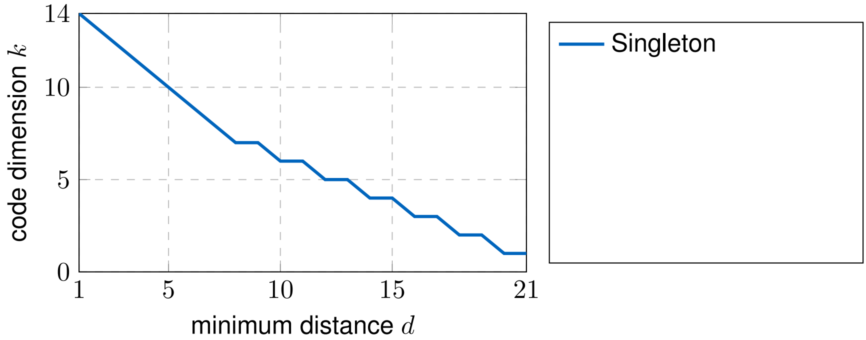


Observation: → MDS codes optimal
→ But smaller field size possible

Comparison of Bounds

$q = 2$, $n_1 = n_2 = 7$, $\lambda_1 = 1$, and $\lambda_2 = 2$



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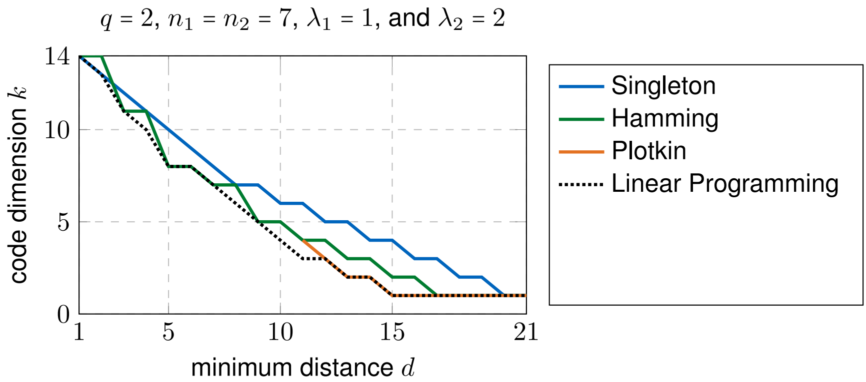

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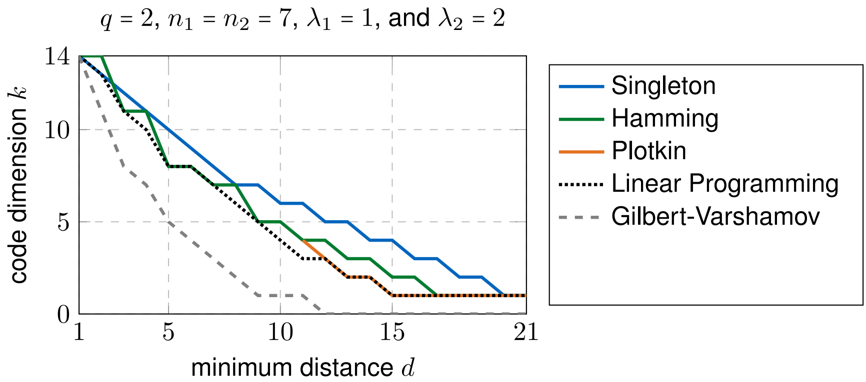

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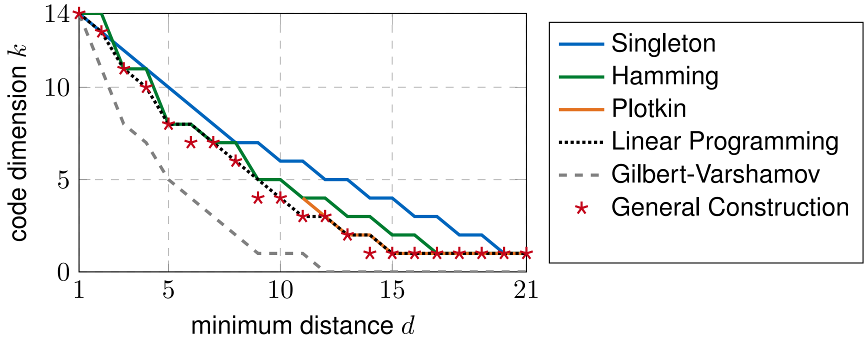

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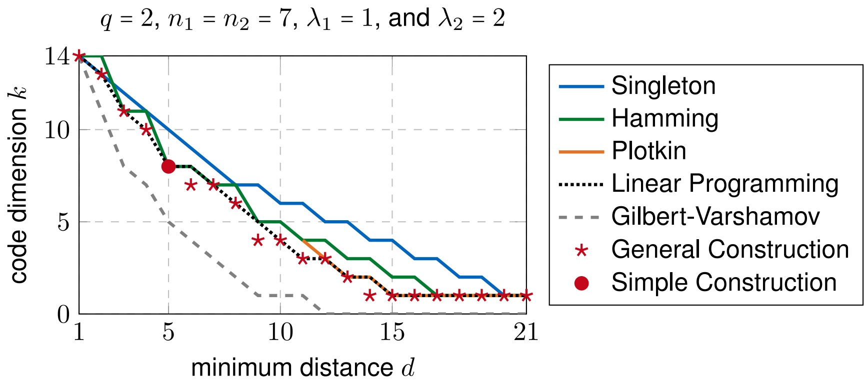
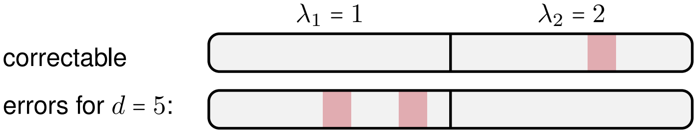


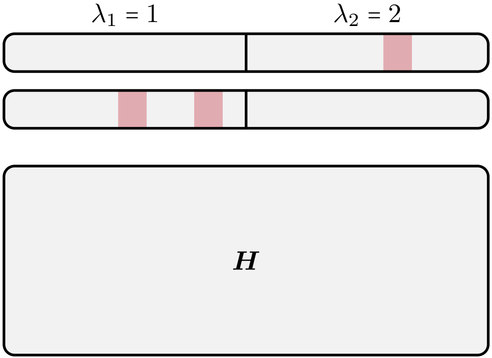
A Simple Construction



A Simple Construction

correctable

errors for $d = 5$:



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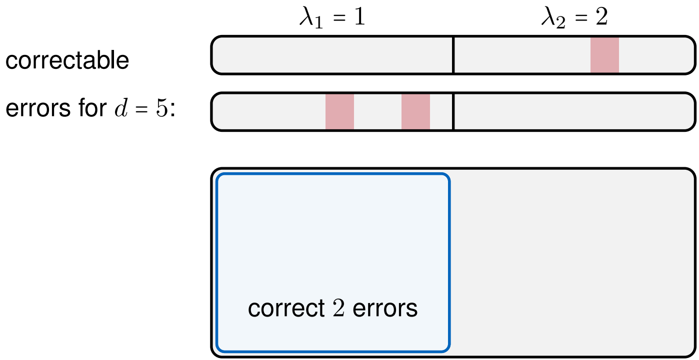

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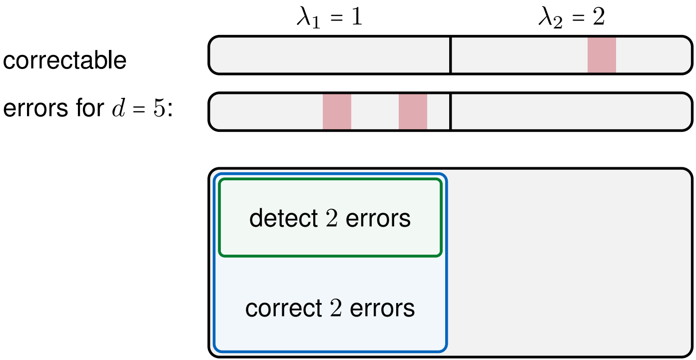

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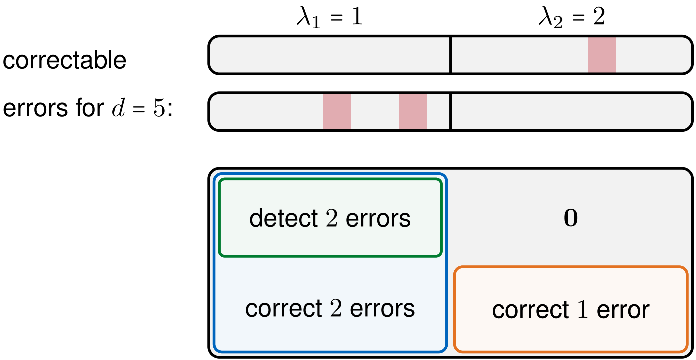

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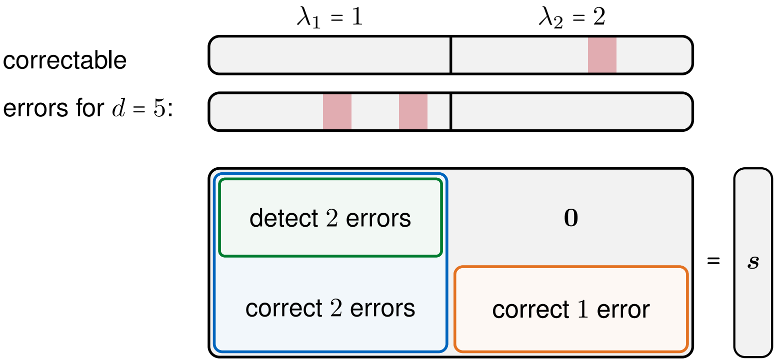

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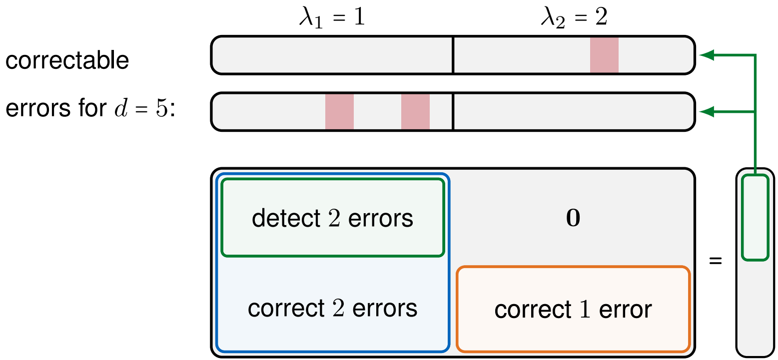

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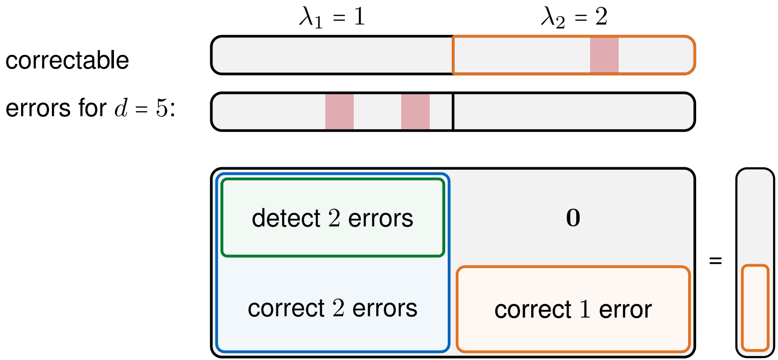

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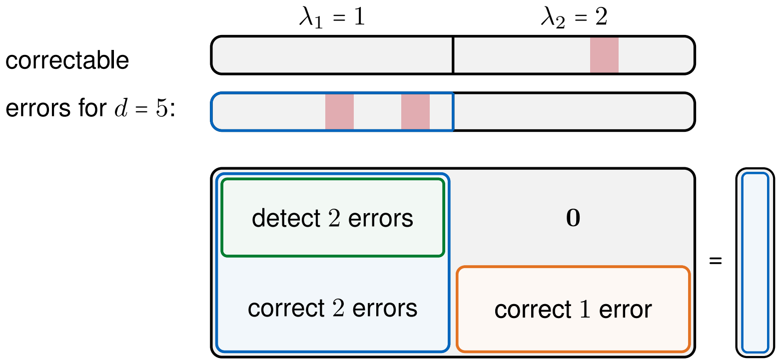


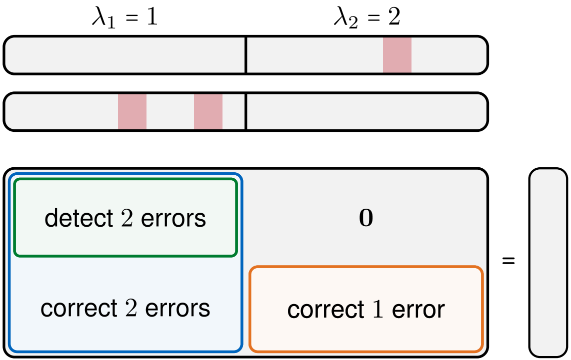
A Simple Construction

correctable

errors for $d = 5$:

optimal redundancy

for various n_1, n_2 and q



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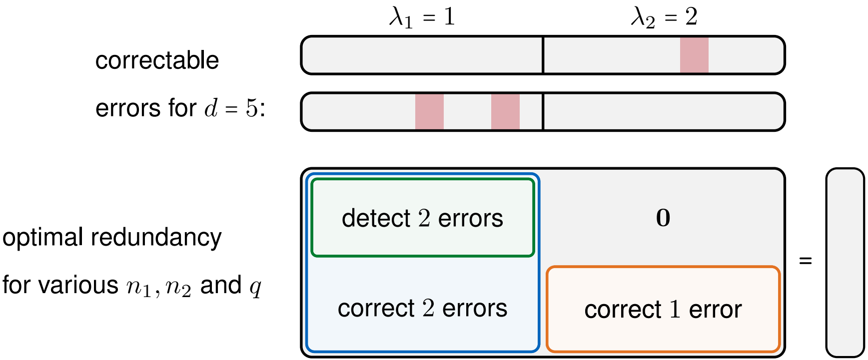

Other d or λ_ℓ: Construction based on generalized code concatenation

Conclusion

arXiv

Weighted-Hamming metric suitable for parallel channels:

- ☺ Error-correction capability can exceed $\left\lfloor \frac{d-1}{2} \right\rfloor$
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Future work:

- ? Bound τ directly instead of d?
- ? Weighted-Hamming d and τ for well-known code classes?

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Thank you!

Questions?