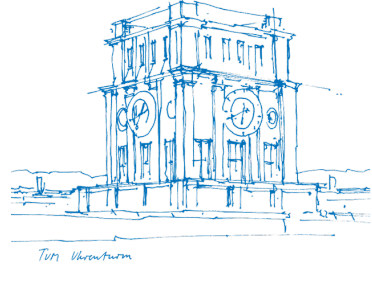
Weighted-Hamming Metric: Bounds and Codes

Sebastian Bitzer
TUM

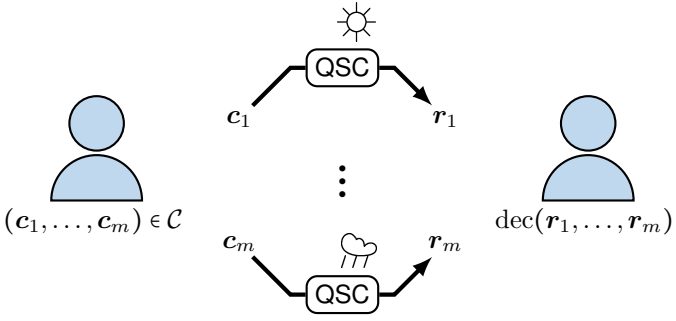
Alberto Ravagnani
TU/e

Violetta Weger
TUM

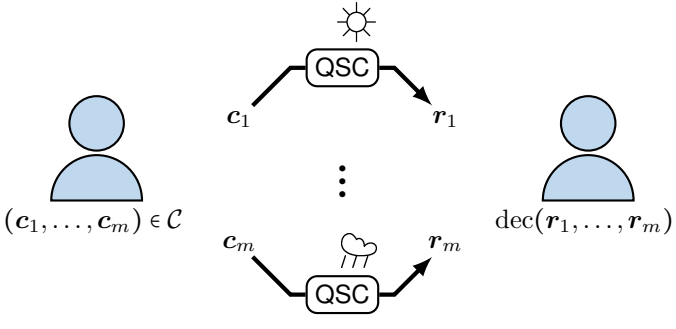
ISIT'26



A Tale of Two Channels



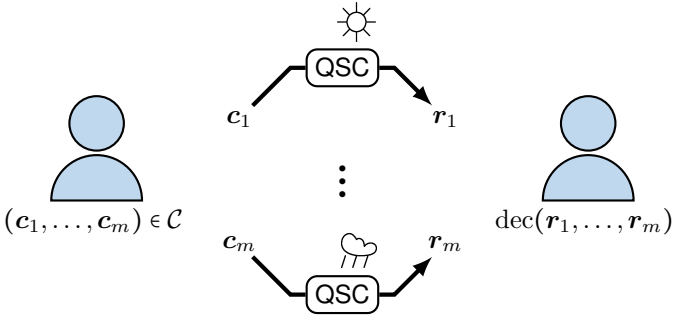
A Tale of Two Channels



Standard solutions:

- Turbo
- LDPC
- Polar
- ...

A Tale of Two Channels



Standard solutions:

- Turbo
- LDPC
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- ...

Goal of this Work

Design short codes with decoding guarantees

From Probability to Metric

Deza, Deza, Encyclopedia of Distances.

Weighted-Hamming Weight

Fix $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$.

$$w_{\boldsymbol{\lambda}}(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m) = \sum_{\ell=1}^{m} \lambda_\ell \cdot w_{\mathrm{H}}(\boldsymbol{c}_\ell)$$

From Probability to Metric

- Deza, Deza, Encyclopedia of Distances.
- B., Ravagnani, Weger, Weighted-Hamming Metric for Parallel Channels

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Matching Parallel QSCs ♡

There exists $\boldsymbol{\lambda} \in \mathbb{N}^m$ s.t.

$$d_{\boldsymbol{\lambda}}(\boldsymbol{c}, \boldsymbol{r}) \leq d_{\boldsymbol{\lambda}}(\boldsymbol{c}', \boldsymbol{r}) \iff \mathsf{Pr}[\boldsymbol{c} \,|\, \boldsymbol{r}] \geq \mathsf{Pr}[\boldsymbol{c}' \,|\, \boldsymbol{r}]$$

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So . . . just maximize the weighted-Hamming distance of $\mathcal{C}$?

Error Correction Done Right

$$\boldsymbol{\lambda} = (1, 2), \quad \mathcal{C} = \{(0, 0, 0 \mid 0, 0, 0),\ (0, 0, 0 \mid 1, 1, 1)\}$$

Error Correction Done Right

$$\boldsymbol{\lambda} = (1, 2), \quad \mathcal{C} = \{(0,\,0,\,0 \mid 0,\,0,\,0),\ (0,\,0,\,0 \mid 1,\,1,\,1)\} \quad \rightarrow \quad d_{\boldsymbol{\lambda}}(\mathcal{C}) = 6$$

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Silva, Kschischang, On Metrics for Error Correction in Network Coding.

Error-Correction Capability

$$t_{\boldsymbol{\lambda}}(\mathcal{C}) = \min_{\boldsymbol{c} \in \mathcal{C} \setminus \{\mathbf{0}\}} t_{\boldsymbol{\lambda}}(\boldsymbol{c})$$

$$t_{\boldsymbol{\lambda}}(\boldsymbol{c}) = \min_{\boldsymbol{r}} \max\{w_{\boldsymbol{\lambda}}(\boldsymbol{r}), w_{\boldsymbol{\lambda}}(\boldsymbol{c}-\boldsymbol{r})\} - 1$$

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Bound on $t_{\boldsymbol{\lambda}}$ via $d_{\boldsymbol{\lambda}}$

Let $\lambda_1 \leq \ldots \leq \lambda_m$.

$$\left\lfloor \frac{d_{\boldsymbol{\lambda}}(\mathcal{C}) - 1}{2} \right\rfloor \leq t_{\boldsymbol{\lambda}}(\mathcal{C}) \leq \left\lfloor \frac{d_{\boldsymbol{\lambda}}(\mathcal{C}) + \lambda_m}{2} \right\rfloor - 1$$

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Previous work:

➔ Minimum distance

Error Correction Done Right

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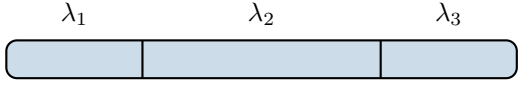
Back to $t_{\boldsymbol{\lambda}}(\mathcal{C})$:

- ➔ Sp. Covering, Singleton, Sp. Packing, LP bounds
- ➔ Concatenated construction

Previous work:

- ➔ Minimum distance

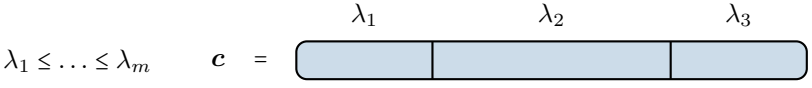
Singleton-Like Bound

$\lambda_1 \leq \ldots \leq \lambda_m$ $\quad \boldsymbol{c} \quad = \quad$ 

$\quad \in \mathcal{C}$

Singleton-Like Bound

$\lambda_1 \leq \ldots \leq \lambda_m$

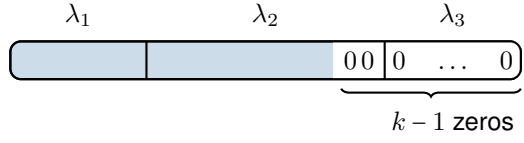


$\in \mathcal{C} = \langle (\boldsymbol{A} \mid \boldsymbol{T}) \rangle$

$\boldsymbol{T}$ lower triangular

Singleton-Like Bound

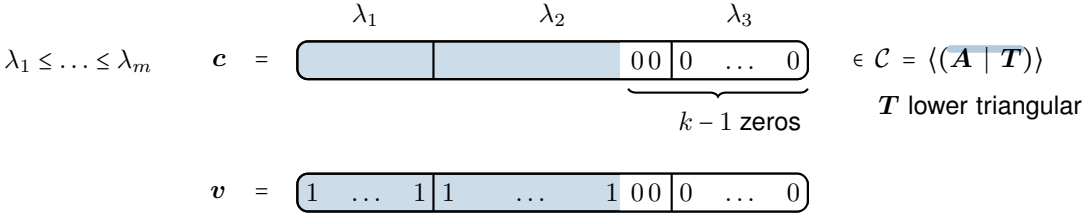
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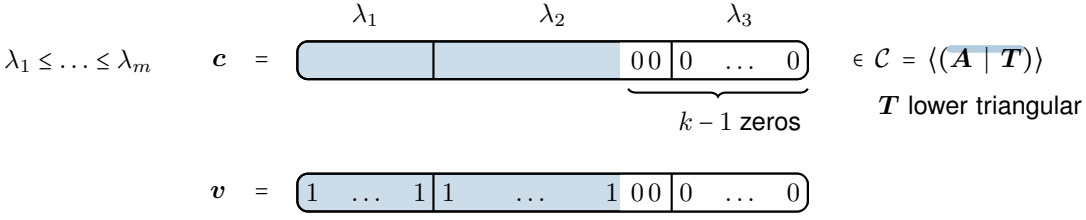
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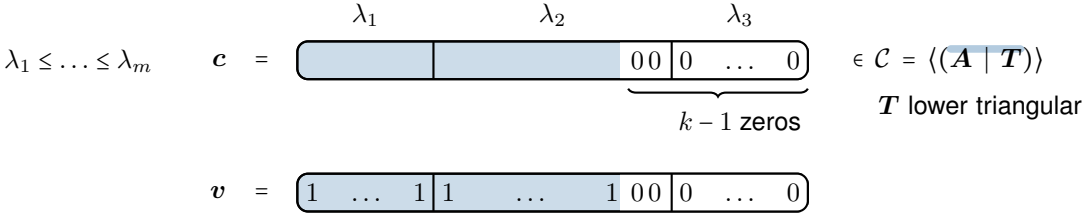
Singleton-Like Bound



Bound on $d_{\boldsymbol{\lambda}}$

$$d_{\boldsymbol{\lambda}}(\mathcal{C}) \leq w_{\boldsymbol{\lambda}}(\boldsymbol{c}) \leq w_{\boldsymbol{\lambda}}(\boldsymbol{v}) \coloneqq d_{\mathrm{SB}}$$

Singleton-Like Bound



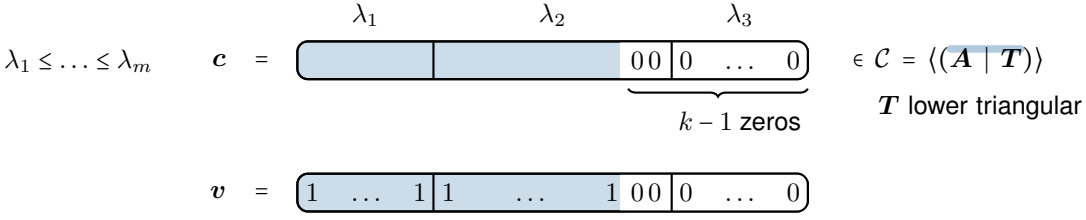
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Bound on $t_{\boldsymbol{\lambda}}$

Naive: $t_{\boldsymbol{\lambda}}(\mathcal{C}) \leq t_{\boldsymbol{\lambda}}(\boldsymbol{c}_{\mathrm{SB}}) \leq \left\lfloor \frac{d_{\mathrm{SB}} + \lambda_m}{2} \right\rfloor - 1$

Singleton-Like Bound



Bound on $d_{\boldsymbol{\lambda}}$

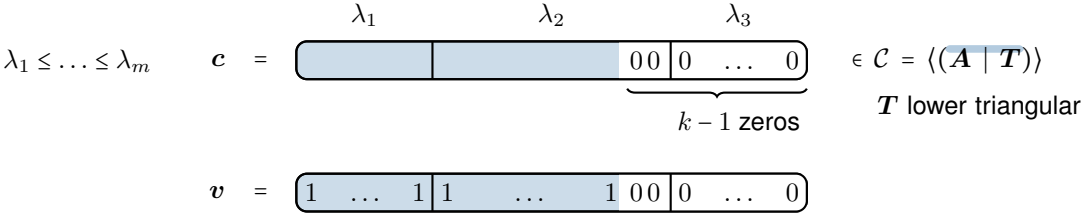
$$d_{\boldsymbol{\lambda}}(\mathcal{C}) \leq w_{\boldsymbol{\lambda}}(\boldsymbol{c}) \leq w_{\boldsymbol{\lambda}}(\boldsymbol{v}) \coloneqq d_{\mathrm{SB}}$$

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Naive: $t_{\boldsymbol{\lambda}}(\mathcal{C}) \leq t_{\boldsymbol{\lambda}}(\boldsymbol{c}_{\mathrm{SB}}) \leq \left\lfloor \frac{d_{\mathrm{SB}} + \lambda_m}{2} \right\rfloor - 1$

Direct: $t_{\boldsymbol{\lambda}}(\mathcal{C}) \leq \left\lfloor \frac{d_{\mathrm{SB}} + \lambda_j}{2} \right\rfloor - 1, \quad j \leq m$

Singleton-Like Bound



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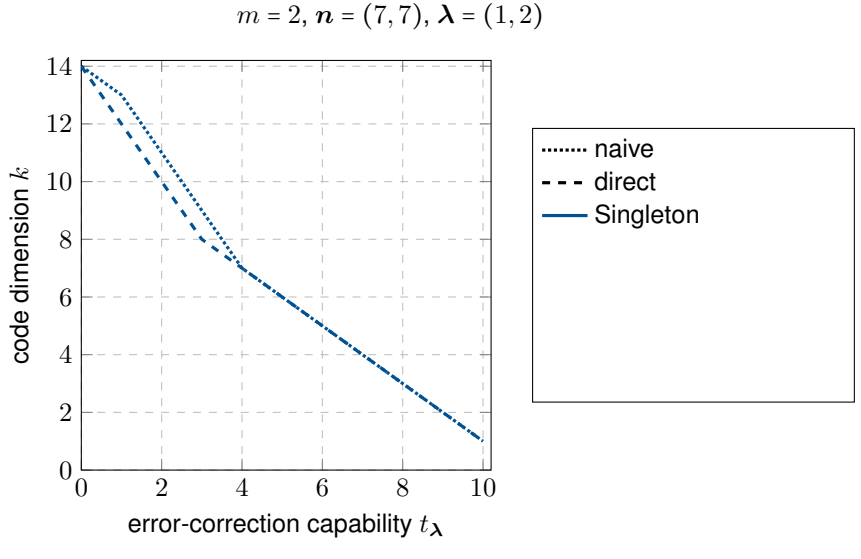
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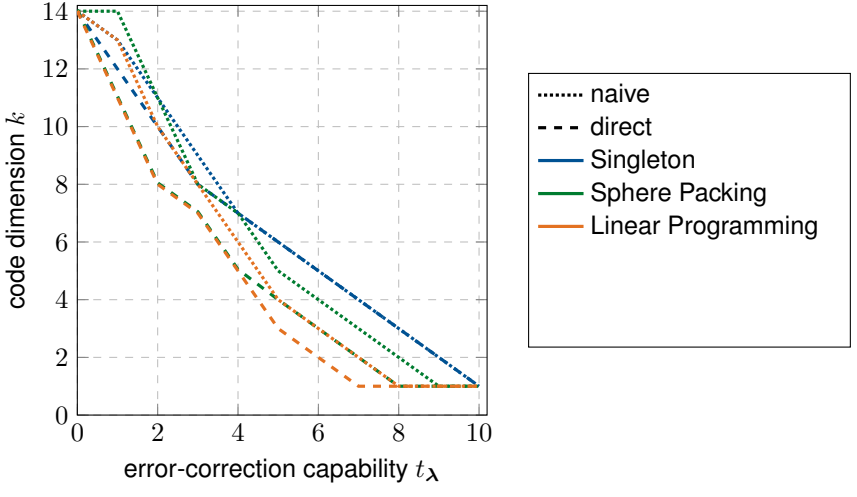
➔ Direct bound tighter

Bounds & Codes



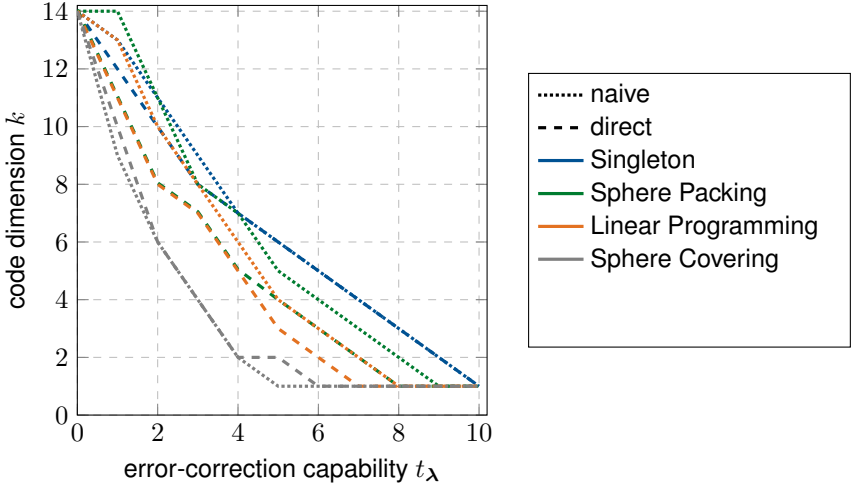
Bounds & Codes

$m = 2$, $\boldsymbol{n} = (7, 7)$, $\boldsymbol{\lambda} = (1, 2)$, $q = 2$



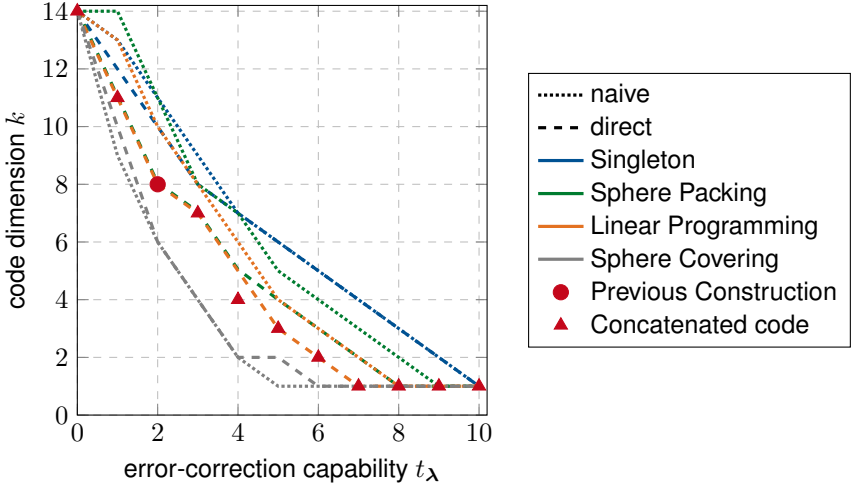
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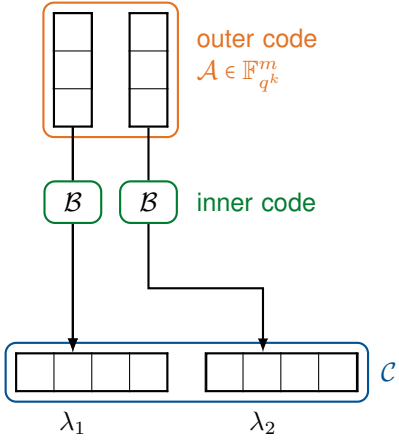


Bounds & Codes

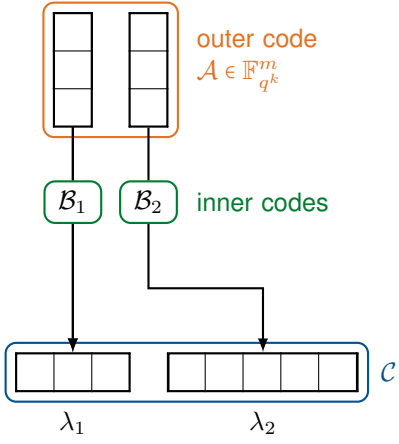
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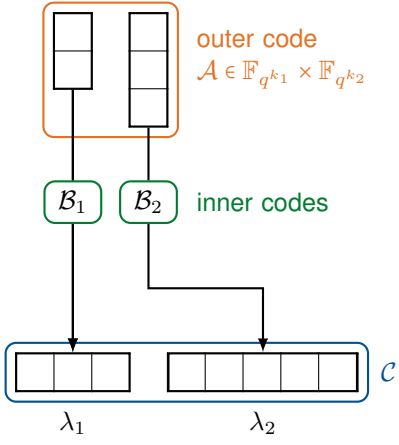
Concatenated Code



Concatenated Code

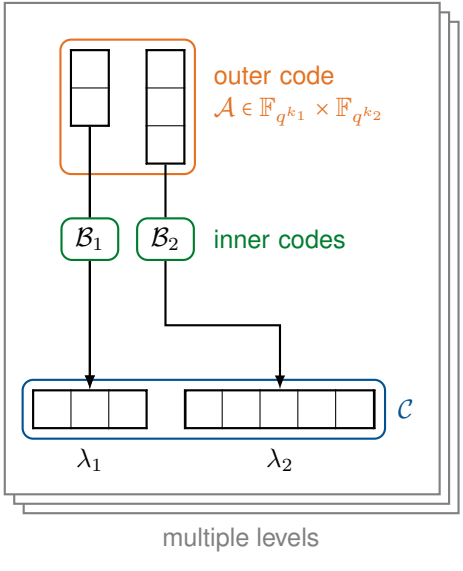


Concatenated Code



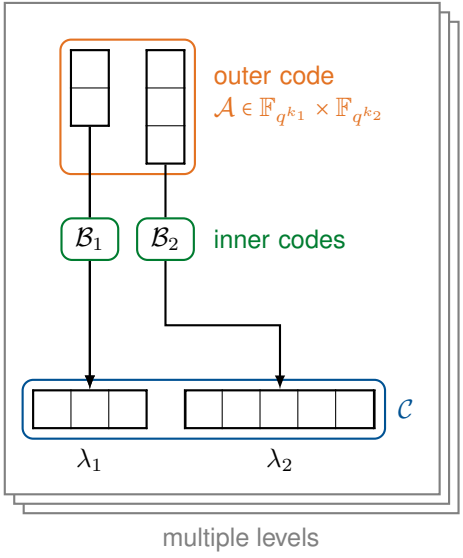
Sidorenko et al., Polyalphabetic Block Codes

Concatenated Code

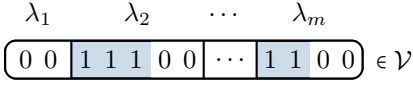


Sidorenko et al., Polyalphabetic Block Codes

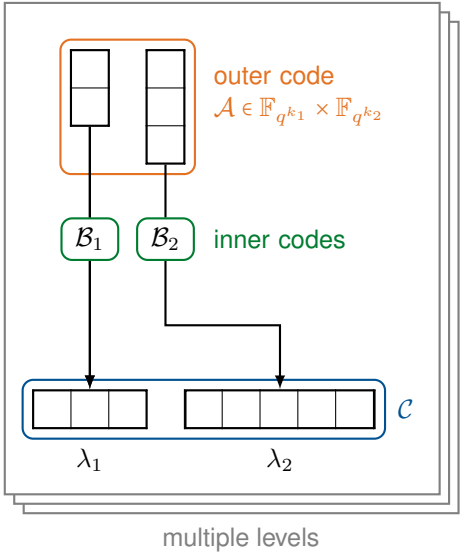
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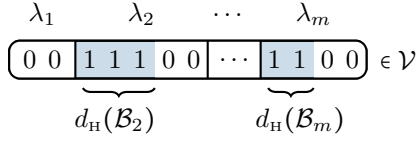
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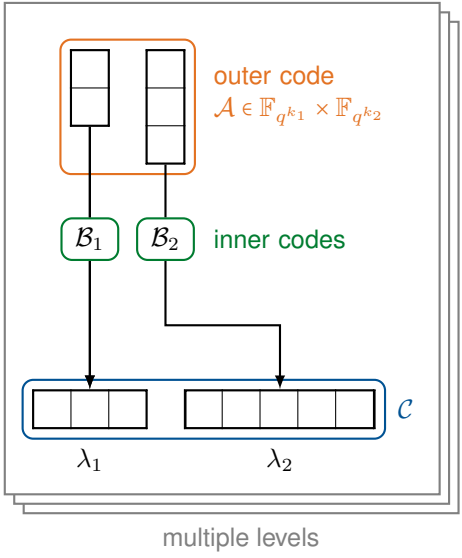
Concatenated Code



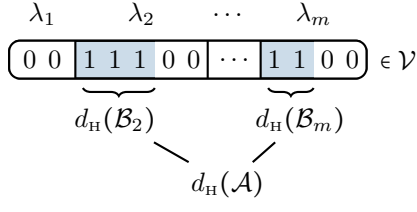
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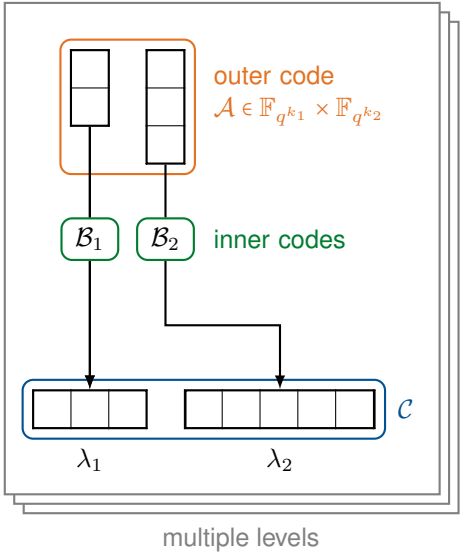
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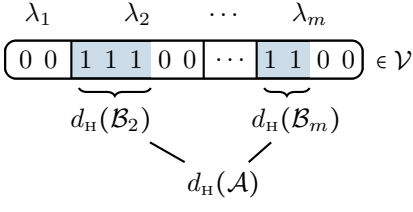
Sidorenko et al., Polyalphabetic Block Codes



Concatenated Code



Sidorenko et al., Polyalphabetic Block Codes

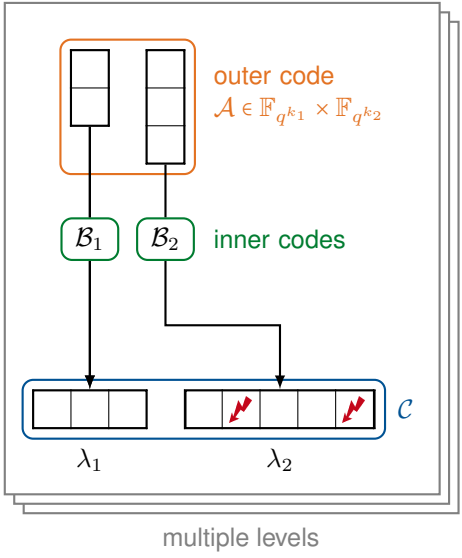


Code Bounds

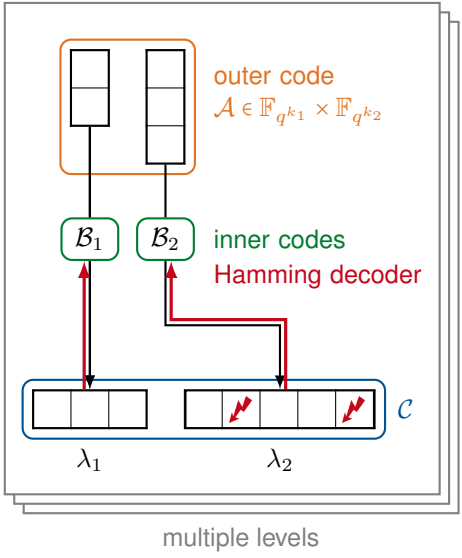
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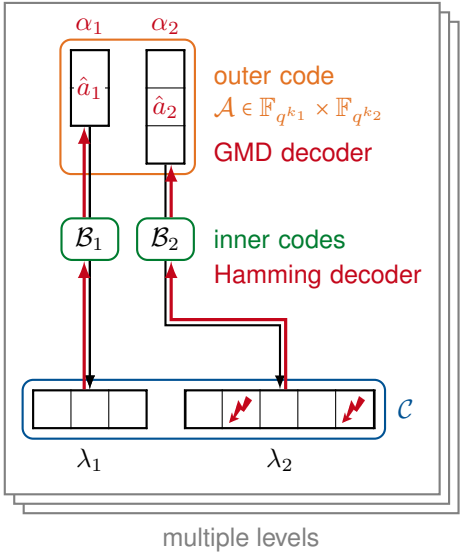
Concatenated Decoder



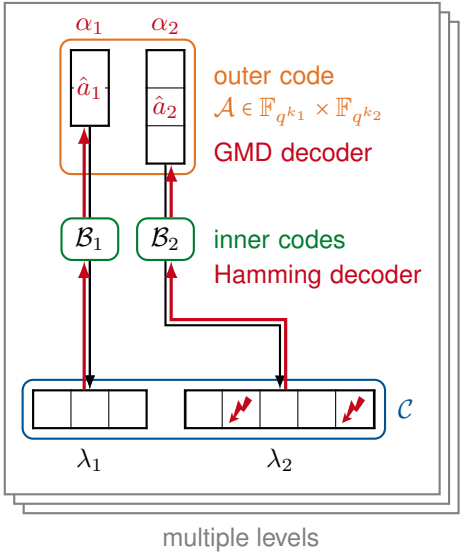
Concatenated Decoder



Concatenated Decoder

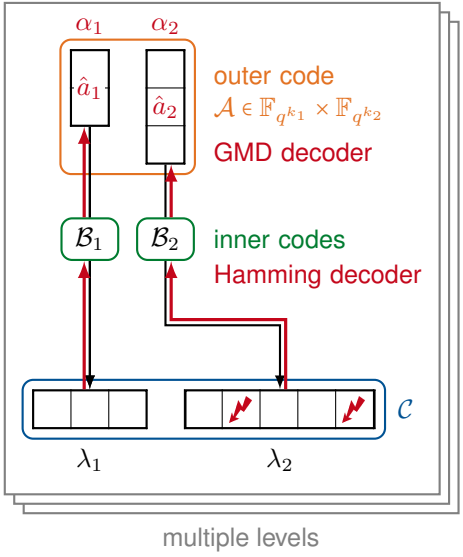


Concatenated Decoder



$$\alpha_\ell = \lambda_\ell \cdot \max\{0,\, d_{\mathrm{H}}(\mathcal{B}_\ell) - 2\, w_{\mathrm{H}}(\hat{\boldsymbol{e}}_\ell)\}$$

Concatenated Decoder



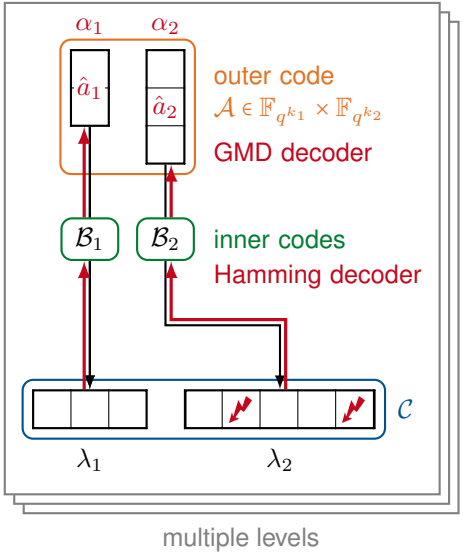
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Taipale, Pursley, An Improvement to Generalized-Minimum-Distance Decoding.

Decoder

$w_{\boldsymbol{\lambda}}(\boldsymbol{e}) \leq t'_{\boldsymbol{\lambda}}(\mathcal{C})$:
$\boldsymbol{e}$ is in the list

Concatenated Decoder



$$\alpha_\ell = \lambda_\ell \cdot \max\{0,\, d_{\mathrm{H}}(\mathcal{B}_\ell) - 2\, w_{\mathrm{H}}(\hat{\boldsymbol{e}}_\ell)\}$$

Taipale, Pursley, An Improvement to Generalized-Minimum-Distance Decoding.

Decoder

$w_{\boldsymbol{\lambda}}(\boldsymbol{e}) \leq t'_{\boldsymbol{\lambda}}(\mathcal{C})$:

$\boldsymbol{e}$ is uniquely decoded

Conclusion

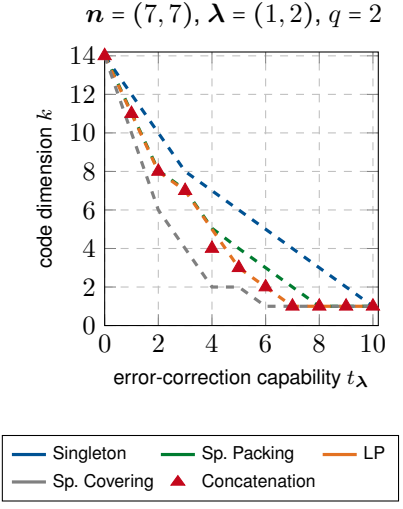
Weighted-Hamming metric:

- ☺ Error-correction capability $t_{\boldsymbol{\lambda}}(\mathcal{C})$ beyond $\left\lfloor \frac{d_{\boldsymbol{\lambda}}(\mathcal{C})-1}{2} \right\rfloor$
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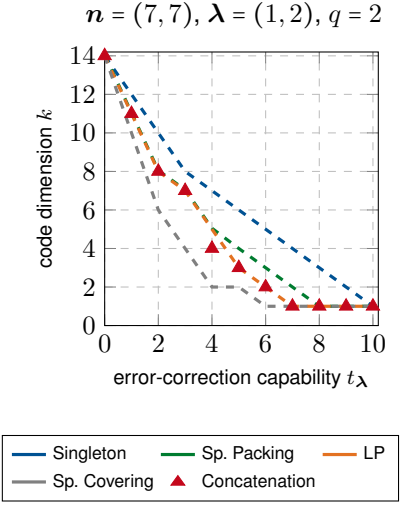
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- Refine characterization of $t_{\boldsymbol{\lambda}}(\boldsymbol{c})$



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Thank you! Questions?

